Membranes for topological M-theory

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ABSTRACT: We formulate a theory of topological membranes on manifolds with G_2 holonomy. The BRST charges of the theories are the superspace Killing vectors (the generators of global supersymmetry) on the background with reduced holonomy $G_2 \subset \mathrm{Spin}(7)$. In the absence of spinning formulations of supermembranes, the starting point is an $N = 2$ target space supersymmetric membrane in seven euclidean dimensions. The reduction of the holonomy group implies a twisting of the rotations in the tangent bundle of the branes with "R-symmetry" rotations in the normal bundle, in contrast to the ordinary spinning formulation of topological strings, where twisting is performed with internal U(1) currents of the $N = (2, 2)$ superconformal algebra. The double dimensional reduction on a circle of the topological membrane gives the strings of the topological A-model (a by-product of this reduction is a Green-Schwarz formulation of topological strings). We conclude that the action is BRST-exact modulo topological terms and fermionic equations of motion. We discuss the rôle of topological membranes in topological M-theory and the relation of our work to recent work by Hitchin and by Dijkgraaf et al.

KEYWORDS: M-Theory, Topological Strings.

Contents

1. Introduction

The notion of topological M-theory was introduced recently by Dijkgraaf et al. [1] (see also [2]). In analogy with topological string theory [8, 9] (for a recent review, see ref. [10]), one expects here a topological membrane world-volume theory to give rise to a field theory in a seven-dimensional target space. In the string case both the world-sheet and the six-dimensional target space theories are fairly well understood, the latter being in fact string field theories constructed from the world-sheet BRST charge. Although Calabi-Yau three-folds have special properties in this context [10], topological strings exist also on special holonomy manifolds of other dimensionalities, see e.g. ref. [3]. The features found in the topological string case would for many reasons be very valuable to understand also in the membrane/M-theory case. One important reason is connected to the rôle topological string amplitudes play in compactification of physical string theories. One may also wonder if a better understanding of topological M-theory may indicate how to approach the problem of finding a microscopic formulation of M-theory, possibly including a quantisation of the membrane.

In ref. [1], the authors took a first step towards this goal by suggesting the form of the effective target space field theory of topological M-theory. Such an effective theory may be obtained by arguing that the theory and its topological properties should be connected to

those of the A and/or B topological string models by dimensional reduction in much the same way as the physical field theories in ten and eleven dimensions are related. Similarly, one should be able to connect the topological string world-sheet theories to the topological membrane one by subjecting the latter to a double dimensional reduction.

The target space aspects were discussed in some detail in ref. [1], where the crucial rôle of Hitchin functionals [5, 6] was elaborated upon. These are special functionals of p-forms which can be connected to metric fields by some rather complicated non-linear relations. The resulting theory was given the appropriate name *form-gravity* in ref. [1]. By starting from a Hitchin 3-form on a seven-dimensional G_2 holonomy manifold, the authors of ref. [1] show that by dimensional reduction various well-known topological form-gravity theories in lower dimensions are obtained. In particular, one finds the Kodaira-Spencer theory [11] for the complex structure deformations in the B-model and the Kähler gravity theory [12] of the Kähler deformations in the A-model, albeit produced in a particular interacting form.

At the classical level the connection between form-gravity based on a six-dimensional Hitchin functional and the topological B-model was made explicit by relating the corresponding tree-level partition functions to each other. However at one loop level, where the B-model is known to compute a special combination of Ray-Singer torsion invariants [11], it was recently demonstrated by Pestun and Witten [15] that one needs to use the extended Hitchin functional introduced in ref. [4] to obtain the same one-loop partition function. This connection to the extended Hitchin functionals is intriguing since they play a rôle also in flux compactifications [20] on the generalised Calabi-Yau manifolds discussed by Hitchin in his paper.

The natural next step seems to be to construct a topological membrane theory that may be related to the topological M-theory mentioned above. That is, we want to construct a membrane embedded in a seven-dimensional space with G_2 holonomy whose effective action is the Hitchin functional 3-form gravity theory discussed in ref. [1]. The usual approach to derive topological strings by means of twisting does not seem to work here since it is based on the spinning string, or NSR, formulation which is lacking in the membrane case. Here we will instead approach this problem by starting from the Green-Schwarz (GS) formulation of the membrane [21]. Of course, since the superstring, and perhaps also the supermembrane, are quantized most easily using Berkovits' pure spinor formulation [22], this is probably an even more suitable starting point. This point was discussed recently also in [23]. We note here that although the GS formulation of string theory is as standard as the NSR one, it does not seem to have been used yet in the construction of topological strings. As will be clear below such a GS formulation will come out of the results presented here for the topological membrane.

One important aspect of twisting in the construction of the topological string from a two-dimensional supersymmetric sigma model is that it turns a spin-$\frac{3}{2}$ supersymmetry current into a spin-1 object that can be interpreted as a BRST current. This kind of twisting is accomplished by enforcing the identification of the world-sheet Lorentz symmetry with an $so(2)$ R-symmetry giving fermionic quantities unphysical integer spin values. In the GS formulation of the membrane, which is the starting point in our approach to the topological membrane, such an unphysical spin-statistics relation on the world-volume is already in

effect since the supercoordinates in the target space $(x^m, \psi^{\hat{\mu}I})$ contain the anticommuting world-volume scalars $\psi^{\hat{\mu}I}$ (the ranges of the various indices will be specified later). For trivial target spaces like eleven-dimensional flat space, the gauge-fixing of the κ-symmetry generates an ordinary supermultiplet in three dimensions with physical spin fermions [21]. However, in the context discussed here no twisting needs to be done by hand, a fact that has been noticed before in ref. [7]. As discussed in detail in section 3, a similar phenomenon to twisting does occur but now as an automatic consequence of combining G_2 holonomy and the tangent space symmetry remaining after the introduction of the membrane into target space. This twisting leaves the bosonic and fermionic fields in the same representation of the surviving symmetry. We will however not fix the gauge, and for the most part work with a fully κ-symmetric theory with a (1+7)-dimensional parameter.

This paper is organised as follows. In section 2 we start by discussing the G_2 3-form gravity theory that the topological membrane is supposed to generate in the seven-dimensional target space. Different action functionals are presented for this theory, one of which we believe is new. This section also describes the supergeometry into which the bosonic seven-dimensional G_2 holonomy manifolds can be embedded. The supercoordinates are $Z^M = (x^m, \theta^{\hat{\mu}I})$ where m runs over seven values and $\hat{\mu}I$ enumerates two ($I = 1, 2$) eight-dimensional spinors ($\hat{\mu} = 1, .., 8$). The supergeometry in encoded by a standard vielbein (supersiebenbein) and a superspace 3-form $C_{MNP}(Z)$. The Bianchi identities are discussed and an explicit 3-form superfield is derived, but only in the flat space limit. As also explained, the full expansions in fermionic coordinates of the curvature dependent 3-form and vielbein superfields can be obtained by a lengthy iterative procedure which we hope to come back to in a future publication (for a similar discussion, see refs. [13, 14]).

In section 3 we discuss the κ-symmetric membrane theory that we propose as the starting point for deriving a topological membrane. The rôle of G_2 in obtaining the BRST charge from a partially gauge fixed world-volume action is explained and arguments indicating the topological nature of the action, namely the fact that it is BRST exact, are presented. This discussion is carried out in the full theory but the calculation of the action is performed only to lowest order in the curvature and a full proof will require more work.

In the concluding section 4 we make a few additional remarks and comments. Properties of the octonions are used heavily in this paper and some aspects can be found in the appendices. In appendix A we discuss G_2 tensors, projection operators and the relation to quaternions, while in appendix B we give the explicit form of the flat superspace 3-form based on the octonionic structure constants.

2. G_2 holonomy

Seven-dimensional manifolds with G_2 holonomy have special properties, among which are Ricci-flatness and a single covariantly constant (Killing) spinor.

When holonomy is restricted to lie in a G_2 subgroup, a (partial) gauge choice can be made for the spin connection to make it lie entirely in the Lie algebra G_2. Then, G_2 singlets can be defined as constant over the manifold, and this thus applies to special elements of any Spin(7) representation containing a G_2 singlet. So, there is a constant spinor, since

$8 \to 1 \oplus 7$, and a constant 3-form Ω, since $35 \to 1 \oplus 7 \oplus 27$. In a flat frame the 3-form may be chosen as $\Omega_{abc} = \sigma_{abc}$, the octonionic structure constants, invariant under the action of G_2, the automorphism group of $\mathbb{O}$ (see appendix A for details).

2.1 The 3-form

Hitchin [5, 6] has constructed a model containing a 3-form field, whose solutions are G_2 manifolds. This is certainly a part of topological M-theory. The metric is constructed from the 3-form as

$$K_{mn} = \sqrt{g}\, g_{mn} = -\frac{1}{144}\varepsilon^{m_1...m_7}\Omega_{mm_1m_2}\Omega_{nm_3m_4}\Omega_{m_5m_6m_7}\,, \tag{2.1}$$

Hitchin gives the action $S = \int d^7x K^{1/9}$. A Polyakov type action, due to Nekrasov [26], giving both the relation (2.1) and the covariant constancy of the 3-form, is

$$S' = \frac{2}{9}\int d^7x \left(\sqrt{g} - \frac{1}{288}g^{mn}\varepsilon^{m_1...m_7}\Omega_{mm_1m_2}\Omega_{nm_3m_4}\Omega_{m_5m_6m_7}\right)\,, \tag{2.2}$$

The metric is auxiliary and determined by its equation of motion. The constant in front is chosen so that the action is normalised to the volume. In a frame where (locally) $\Omega_{abc} = \sigma_{abc}$, one thus has $g_{ab} = \delta_{ab}$, which is checked by $\sigma_{acd}\sigma_{bef}\star\sigma_{cdef} = -24\delta_{ab}$ (see appendix A).

Varying the action (2.2) w.r.t. Ω gives

$$-\tfrac{1}{9\times144}\int d^7x (\; 2g^{rn}\varepsilon^{stm_3...m_7}\Omega_{nm_3m_4}\Omega_{m_5m_6m_7} \\ +g^{mn}\varepsilon^{m_1...m_4rst}\Omega_{mm_1m_2}\Omega_{nm_3m_4})\delta\Omega_{rst}\,. \tag{2.3}$$

Using the relations of appendix A to calculate the two terms, one finds using the expression for g_{mn} that they both are proportional to the same expression, and that the variation (2.3) becomes

$$\frac{1}{3}\int \star\Omega \wedge \delta\Omega = \frac{1}{18}\int d^7x\sqrt{g}\,\Omega^{mnp}\delta\Omega_{mnp}\,. \tag{2.4}$$

The relation (2.1) for the metric may equivalently be written in the implicit form

$$g_{mn} = \frac{1}{6}g^{p_1q_1}g^{p_2q_2}\Omega_{mp_1p_2}\Omega_{nq_1q_2}\,, \tag{2.5}$$

which is used by Hitchin in expressing the variation of his action in the "linear form" (2.4). This latter relation could as well be obtained from an action, which now takes a much more conventional form:

$$S'' = -\frac{1}{6}\int d^7x\sqrt{g}\left(1 - \frac{1}{6}g^{m_1n_1}g^{m_2n_2}g^{m_3n_3}\Omega_{m_1m_2m_3}\Omega_{n_1n_2n_3}\right) \tag{2.6}$$

(what varying this action w.r.t. g^{mn} really gives is $\Omega_{mpq}\Omega_n{}^{pq} = -g_{mn}(1 - \frac{1}{6}\Omega_{pqr}\Omega^{pqr})$, which after contracting the free indices with g^{mn} gives $\Omega_{pqr}\Omega^{pqr} = 42$, and thus $g_{mn} = \frac{1}{6}\Omega_{mpq}\Omega_n{}^{pq}$). Variation of the action (2.6) w.r.t. Ω gives an expression proportional to (2.4) directly, without any use of the algebraic identities of appendix A.

The 3-form is part of the geometric background for propagation of membranes. The expression "$\Omega = \sigma$" is purely bosonic. In a superspace, Ω will contain more components when expressed in flat basis, due to torsion (see appendix B).

2.2 Superspace and supersymmetry

The superspace we want to consider has bosonic coordinates which are the coordinates of a euclidean manifold with G_2 holonomy. In addition there will be fermionic coordinates. These are *a priori* a set of real spinors in the 8-dimensional representation of Spin(7), but when Spin(7) $\rightarrow G_2$ each spinor decomposes as $8 \rightarrow 1 \oplus 7$. The γ-matrices of Spin(7) are real and antisymmetric, so it is clear that an even number of spinors are needed, together with an internal Sp($2n$) in order to have a non-vanishing torsion. We will choose the simplest possibility, $n = 1$, giving a doublet of spinors, for reasons that become obvious in the following subsection. This superspace is obtained from $D = 11$ superspace, with twice as many fermionic coordinates, as a truncation of the Spin(7) $\times$ SL(2, $\mathbb{C}$) subgroup of Spin(1, 10) to Spin(7) $\times$ SL(2, $\mathbb{R}$), where the spinors in the representation $32 \rightarrow (8, 2_{\mathbb{C}})$ are demanded to be real.

A convenient realisation is to consider a vector as an imaginary octonion, $v \in \mathbb{O}'$, and a spinor as an arbitrary octonion, $s \in \mathbb{O}$. Letting the orthonormal basis of $\mathbb{O}'$ be $\{e_a\}_{a=1}^7$, multiplication by γ^a is identified with left multiplication of a spinor with e_a, i.e., vs is again a spinor. The octonionic multiplication table, $e_a e_b = -\delta_{ab} + \sigma_{abc} e_c$, tells us that the real γ-matrices square to -1 (a property which will be crucial for supermembranes).

Before moving on let us fix some notation. Superspace coordinates are written

$$Z^M = (x^m, \psi^{\hat{\mu}I}), \quad m = 1, \ldots 7, \quad \hat{\mu} = 0, \ldots, 7, \quad I = 1, 2. \tag{2.7}$$

Flat indices are written $(a, \hat{\alpha}I)$. The spinor index will often be divided into $(0, \alpha = 1, \ldots, 7)$, reflecting the decomposition $8 \rightarrow 1 \oplus 7$. This division applies also to curved indices, as long as one only considers super-diffeomorphisms that leave the singlet inert, and we use the notation

$$\psi^{\hat{\mu}I} = (\theta^I, \psi^{\mu I}). \tag{2.8}$$

Bosonic and fermionic vielbeins are written,

$$E^a, \quad \mathcal{E}^{\hat{\alpha}I} = (\mathcal{E}^I, \mathcal{E}^{\alpha I}), \tag{2.9}$$

and the purely bosonic vielbein, $e_m{}^a$.

The γ matrices encoded in the left multiplication of a spinor $\lambda = \lambda^{\hat{\alpha}} e_{\hat{\alpha}}$ by an imaginary unit e_a are

$$\begin{aligned}
(\gamma^a)_{\alpha\beta} &= \sigma^a{}_{\alpha\beta}, \\
(\gamma^a)_{0\alpha} &= \delta^a_\alpha.
\end{aligned} \tag{2.10}$$

They satisfy $\{\gamma^a, \gamma^b\} = -2\delta^{ab}$, where the minus sign is necessary for real γ-matrices.

The Clifford algebra is spanned by the $so(7)$-invariant tensors $\delta^{\hat{\alpha}}{}_{\hat{\beta}}$, $(\gamma^a)^{\hat{\alpha}}{}_{\hat{\beta}}$, $(\gamma^{ab})^{\hat{\alpha}}{}_{\hat{\beta}}$ and $(\gamma^{abc})^{\hat{\alpha}}{}_{\hat{\beta}}$, of which the first and last are symmetric and the second and third antisymmetric

matrices. The decomposition in terms of G_2-invariant tensors is

$$
\begin{aligned}
\delta^{\hat{\alpha}}{}_{\hat{\beta}} &= \begin{bmatrix} 1 & 0 \\ 0 & \delta^{\alpha}{}_{\beta} \end{bmatrix} \\
(\gamma^a)^{\hat{\alpha}}{}_{\hat{\beta}} &= \begin{bmatrix} 0 & \delta^a{}_{\beta} \\ -\delta^{a\alpha} & \sigma^{a\alpha}{}_{\beta} \end{bmatrix} \\
(\gamma^{ab})^{\hat{\alpha}}{}_{\hat{\beta}} &= \begin{bmatrix} 0 & -\sigma^{ab}{}_{\beta} \\ \sigma^{ab\alpha} & -\star\sigma^{ab\alpha}{}_{\beta} - 2\delta^{ab}_{\alpha\beta} \end{bmatrix} \\
(\gamma^{abc})^{\hat{\alpha}}{}_{\hat{\beta}} &= \begin{bmatrix} \sigma^{abc} & -\star\sigma^{abc}{}_{\beta} \\ -\star\sigma^{abc\alpha} & 6\delta^{[a}_{(\alpha}\sigma_{\beta)}{}^{bc]} - \delta^{\alpha}{}_{\beta}\sigma^{abc} \end{bmatrix}.
\end{aligned}
\tag{2.11}
$$

Solving the dimension-0 part of the torsion Bianchi identities reveals a possible solution in terms of SO(7) γ-matrices. We choose $T^a_{\alpha I, \beta J} = 2\varepsilon_{IJ}(\gamma^a)_{\alpha\beta}$, implying

$$
\begin{aligned}
T_{\alpha I, \beta J}{}^a &= 2\varepsilon_{IJ}\sigma^a{}_{\alpha\beta}, \\
T_{0I, \alpha J}{}^a &= 2\varepsilon_{IJ}\delta^a_{\alpha}, \\
T_{0I, 0J}{}^a &= 0.
\end{aligned}
\tag{2.12}
$$

(A wider class of G_2-invariant solutions exists but κ-symmetry restricts the choice to eq. (2.12).) The background will contain a 3-form potential C (descending from the one in $D = 11$) with 4-form field strength, G, whose dimension-0 part is taken to be $G_{ab,\hat{\alpha}I,\hat{\beta}J} = -2\varepsilon_{IJ}(\gamma_{ab})_{\hat{\alpha}\hat{\beta}}$:

$$
\begin{aligned}
G_{ab,\alpha I, \beta J} &= 2\varepsilon_{IJ}(2\delta^{ab}_{\alpha\beta} + \star\sigma_{ab\alpha\beta}), \\
G_{ab,0I, \beta J} &= 2\varepsilon_{IJ}\sigma_{ab\beta}, \\
G_{ab,0I, 0J} &= 0.
\end{aligned}
\tag{2.13}
$$

The Fierz identity in $D = 7$ ensuring the Bianchi identity for G is

$$
(\gamma^b)_{\hat{\alpha}\hat{\beta}}(\gamma_{ab})_{\hat{\gamma}\hat{\delta}}|_{\boxplus} = 0,
\tag{2.14}
$$

where the Young tableau indicates the symmetry structure of the spinor indices. The expression $(\gamma^b)_{\hat{\alpha}\hat{\beta}}(\gamma_{ab})_{\hat{\gamma}\hat{\delta}}$ contains only terms that are antisymmetric in at least three spinor indices, implying that $\varepsilon_{IJ}\varepsilon_{KL}(\gamma^b)_{\hat{\alpha}\hat{\beta}}(\gamma_{ab})_{\hat{\gamma}\hat{\delta}}$ completely symmetrised in the four composite indices $(\hat{\alpha}I, \hat{\beta}J, \hat{\gamma}K, \hat{\delta}L)$ vanishes.

The potential C, which will be the field that the supermembranes couples minimally to, is *a priori* thought of as a 3-form with vanishing cohomology class, so that, modulo gauge transformations, $C_{abc} = 0$. Of course, changing C to $C^{(k)} = C + k\Omega$ leaves the field strength invariant.

The constraints for torsion and field strength used are standard, and the ones obtained by reduction from $D = 11$ and truncation to real fermions. In order to use them to extract an explicit form for the dynamics of the supermembrane introduced in the following section, one would need to solve these constraints explicitly for the vielbeins and components of C in terms of the bosonic and fermionic coordinates. This has not been done, except for in the case of flat manifolds (orbifolds of tori). In principle, this can be done order by order in the fermions, and we will indicate how this expansion starts.

The target space coordinates[1] are x^m, ψ^{mI} and θ^I. Under (bosonic) diffeomorphisms, $\delta_\chi x^m = \chi^m$, $\delta_\chi \psi^{mI} = \psi^{nI} \partial_n \chi^m$, $\delta_\chi \theta^I = 0$. This means that the derivatives and dual differentials that transform covariantly are

$$
\begin{aligned}
& dx^m && \mathcal{D}_m = \partial_m - \Gamma^p_{mn} \psi^{nI} \frac{\partial}{\partial \psi^{pI}} \\
& D\psi^{mI} = d\psi^{mI} + dx^n \psi^{pI} \Gamma^m_{np} && \frac{\partial}{\partial \psi^{mI}} \\
& d\theta^I && \frac{\partial}{\partial \theta^I}
\end{aligned}
\tag{2.15}
$$

(if we have differentials that transform covariantly, we can just contract them with $e_m{}^a$ to get something that is invariant). In order to reproduce the dimension-0 torsion, the vielbeins are constructed from the covariant differentials as

$$
\begin{aligned}
E^a &= (dx^m + \varepsilon_{IJ} \Omega^m{}_{np} D\psi^{nI} \psi^{pJ} + 2\varepsilon_{IJ} d\theta^I \psi^{mJ}) e_m{}^a + \dots, \\
\mathcal{E}^{aI} &= D\psi^{mI} e_m{}^a + \dots, \\
\mathcal{E}^I &= d\theta^I.
\end{aligned}
\tag{2.16}
$$

We also let $\omega = dx^m \omega_m(x)$ and $D = d + \omega$. These terms generate torsion, however, which contains the Riemann tensor ($\mathcal{T}^{aI} \equiv T^{\alpha I} \delta^a_\alpha$):

$$
\begin{aligned}
T^a &= \varepsilon_{IJ} (\Omega^m{}_{np} D\psi^{nI} \wedge D\psi^{pJ} + 2 d\theta^I \wedge D\psi^{mJ}) e_m{}^a + \Omega^m{}_{np} R_q{}^n \psi^{qI} \psi^{pJ} e_m{}^a, \\
\mathcal{T}^{aI} &= \psi^{nI} R_n{}^m e_m{}^a,
\end{aligned}
\tag{2.17}
$$

where $D\Omega = 0$ has been used. The curvature enters with $R_m{}^n \equiv \frac{1}{2} dx^p \wedge dx^q R_{qpm}{}^n$. So, while the correct torsion terms are generated, the curvature-dependent ones have to be compensated for by adding terms of higher order in fermions in the vielbeins. Note, however, that this does not apply to the coefficients of $d\theta^I$, which is a G_2 singlet, hence not affected by the spin connection, and furthermore exact.

2.3 G_2 manifolds and supersymmetry

The existence of a constant spinor allows for a Killing spinor, a fermionic "isometry" of the superspace, i.e., a global supersymmetry. In our superspace with an internal SL(2) index, there will be a doublet of supersymmetries. We choose a parametrisation where the superspace Killing vectors, i.e., the supersymmetry generators are

$$
Q_I = \frac{\partial}{\partial \theta^I},
\tag{2.18}
$$

which obviously fulfill

$$
\{Q_I, Q_J\} = 0.
\tag{2.19}
$$

All vielbeins in eq. (2.16) are invariant under Q_I. We may remark that the simple form (2.18) of the supersymmetry generators depends on the the form of the bosonic vielbeins. If the fermion bilinears in the bosonic vielbein had been chosen to contain $\varepsilon_{IJ}(d\theta^I \psi^{mJ} - \theta^I d\psi^{mJ})$ instead of $2\varepsilon_{IJ} d\theta^I \psi^{mJ}$ (which to lowest order corresponds to a

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[1]The identification of part of the spinor as vectors involves gauge-fixing all except the bosonic diffeomorphisms.

change of bosonic coordinate), one would also have had a term $-\varepsilon_{IJ}\psi^{mI}\frac{\partial}{\partial x^m}$ in Q_I. Diffeomorphism covariance would then demand that $\frac{\partial}{\partial x^m}$ is replaced by $\mathcal{D}_m$, so also ψ is transformed. It turns out (by trial and error) that it is impossible to construct a supersymmetry doublet that starts out this way and fulfills the nilpotency relation (2.19), due to curvature terms, so we are left with the choice of eq. (2.18).

In the discussion on a topological theory of membranes below, the Q_I's are the nilpotent operators that will be promoted to BRST operators.

3. Topological membranes

In this section, we will describe in detail how we obtain a topological membrane by imposing a supersymmetry constraint on supermembranes embedded in a superspace extending a manifold of G_2 holonomy. First we will introduce supermembranes, and investigate the structure of κ-symmetry in the background at hand. We proceed to promote the global supersymmetry generators to BRST operators, thereby turning the theory into a topological theory. We show that the action, modulo topological terms and fermionic equations of motion, is not only BRST-invariant, but also BRST-exact.

3.1 Supermembranes on G_2 manifolds

A supermembrane in seven dimensions should have $N = 2$ supersymmetry, i.e., propagate in a background superspace with two fermionic spinorial coordinates $\psi^{\hat{\mu}I}$. Then the four transverse bosons match the fermions in number, with κ-symmetry and equations of motion taken into account. Both bosons and fermions are *a priori* scalars on the world-volume. This can of course change after some gauge-fixing, e.g. choosing a static gauge. The superspace we choose for the propagation of the membrane is thus taken to be the one described in the previous section.

When formulating a theory of topological *strings* it is convenient to start from the action of a spinning (world-sheet supersymmetric) string. For a membrane, no such formulation exists that is equivalent to the space-time supersymmetric one. Since we want our membrane to describe part of M-theory, we seem to be forced to use the ordinary supermembrane action. The generic action for a supersymmetric membrane is

$$S = \int d^3\xi \sqrt{g} + \int C\,. \tag{3.1}$$

where g and C are pullbacks from target superspace to the world-volume.

The 7-dimensional R-symmetry is SL(2). R-symmetry is typically something one wants to use in a topological twist, but the real forms of R-symmetry and local world-volume rotations $su(2)$ do not match. On the other hand, once one decomposes rotations into longitudinal and transverse, there are lots of $su(2)$'s. When $so(7) \rightarrow so(3) \oplus so(4) \approx su(2) \oplus su(2) \oplus su(2)$, $7 \rightarrow (1,2,2) \oplus (3,1,1)$ and $8 \rightarrow (2,1,2) \oplus (2,2,1)$. But if we also have the breaking $so(7) \rightarrow G_2$, $7 \rightarrow 7$, $8 \rightarrow 1 \oplus 7$, we have to consider the maximal unbroken subalgebra contained in both G_2 and $su(2)^3$. In the case that the embedding of the membrane world-volume is associative, i.e., if $\star\sigma_{ijka} = 0$, or equivalently $\sigma_{ijk} = \pm\varepsilon_{ijk}$,

this is $su(2) \oplus su(2)$, which is a maximal subalgebra of G_2, and where the second $su(2)$ is the last of the three in $so(7) \rightarrow su(2) \oplus su(2) \oplus su(2)$ and the first is the diagonal subalgebra of the first two (this is shown in detail in appendix A, using the splitting of an octonion into a pair of quaternions). For a more general embedding, the same representations are obtained in a static gauge based on coordinate directions spanning a quaternion.

From a 3-dimensional perspective, we have (before G_2 is imposed) scalars transforming as vectors under R-symmetry $so(4)$, $\phi \in (1, 2, 2)$, and spinors transforming as either of the chiralities of $so(4)$, $\psi \in (2, 2, 1)$ and/or $\psi' \in (2, 1, 2)$. *Introduction of G_2 implies a twist of one of the spinor representations*, since it identifies one of the two R-symmetry $su(2)$'s with the $su(2)$ of space rotations. This twisting has been observed earlier in ref. [7].

The lesson from the behaviour of the representations and the effective twisting is that when one wants to formulate a topological membrane theory, no twisting "by hand" is needed — it is automatically provided in a space-time supersymmetric formulation.

3.2 Fermionic symmetries

The supermembrane action is invariant under global supersymmetry as well as κ-symmetry. Let us discuss these symmetries in some more detail, beginning with supersymmetry, generated by the vector fields $Q_\varepsilon = \varepsilon^I \frac{\partial}{\partial \theta^I}$, with constant parameters ε^I.

All vielbeins, both the bosonic ones E^a and the fermionic ones $\mathcal{E}^{\hat\alpha I} = (d\theta^I, \mathcal{E}^{\alpha I})$, are invariant under supersymmetry — this is just the statement that supersymmetry is an isometry of superspace. This accounts for the invariance of the kinetic volume term in the supermembrane action.

Invariance of the Wess-Zumino term $\int C$ is guaranteed by the invariance of the field strength G of eq. (2.13). The field strength is expressible as constant coefficients times wedge products of vielbeins, and thus invariant. This implies that the supersymmetry transformation of C is a total derivative, $Q_\varepsilon C = \varepsilon^I d\Lambda_I$. It is indeed possible to choose a gauge where a stronger statement, namely local invariance, $Q_\varepsilon C = 0$, holds. We have constructed C explicitly in such gauges (to lowest order in curvature), see appendix B. The fact that C can be chosen to be completely independent of θ^I will later, when Q_I are used as BRST operators, be a crucial property.

To begin our exposé of κ-symmetry for the topological membrane we recount some well known facts concerning its properties. In order to reduce clutter we drop the $sl(2)$-indices temporarily, reinserting them when returning to the topological membrane. We begin by introducing the superspace vector field,

$$\kappa = \kappa^M \partial_M = \kappa^\alpha \mathcal{E}_\alpha^M \partial_M \quad (\kappa^a = 0), \tag{3.2}$$

the action of which transforms the pullback of a superspace form as,

$$\delta_\kappa(f^*\Omega) = f^* \mathcal{L}_\kappa \Omega = f^*(i_\kappa d + d i_\kappa)\Omega, \tag{3.3}$$

where f^* is a pullback and $\mathcal{L}$ a Lie derivative. From here on we will not write out pullbacks explicitly. The action of this vector field on the Wess-Zumino term then follows,

$$\delta_\kappa \int C = \int (i_\kappa d + d i_\kappa) C = \int (i_\kappa G + d i_\kappa C) = \int i_\kappa G, \tag{3.4}$$

and the variation of the vielbein,

$$
\begin{aligned}
\delta_\kappa E^A &= i_\kappa(T^A - E^B \wedge \omega_B{}^A) + Di_\kappa E^A - (i_\kappa E^B) \wedge \omega_B{}^A \\
&= i_\kappa T^A - E^B \wedge i_\kappa \omega_B{}^A + Di_\kappa E^A \,.
\end{aligned}
\tag{3.5}
$$

By adding a local Lorentz transformation with parameter $i_\kappa \omega_B{}^A$, we can reduce the expression to $\delta_\kappa E^A = i_\kappa T^A + Di_\kappa E^A$, and furthermore, by considering the relevant part of this expression, to

$$
\delta_\kappa E^a = i_\kappa T^a \,.
\tag{3.6}
$$

The variation of the kinetic term then becomes (with pullbacks written out)

$$
\delta_\kappa \sqrt{g} = \frac{1}{2}\sqrt{g}g^{ij}\delta_\kappa g_{ij} = \sqrt{g}g^{ij}E^a_{(i}E^B_{j)}\kappa^\alpha T_{\alpha B}{}^a \,,
\tag{3.7}
$$

where we have used $\delta_\kappa g_{ij} = \delta_\kappa(E^a_i E^a_j) = 2E^a_{(i}E^B_{j)}\kappa^\alpha T_{\alpha B}{}^a$. At the level of (length-)dimension 0 this term varies as

$$
\delta_\kappa \sqrt{g} = \sqrt{g}g^{ij}E^a_i E^\beta_j \kappa^\alpha T_{\alpha\beta}{}^a = \sqrt{g}E^\beta_j T_{\alpha\beta}{}^j \kappa^\alpha \,,
\tag{3.8}
$$

whereupon the action consequently transforms as

$$
\delta_\kappa \left(\int d^3\xi \sqrt{g} + \int C \right) = \int d^3\xi \sqrt{g} \left(E^\beta_i T_{\alpha\beta}{}^i \kappa^\alpha + \frac{1}{2}\frac{\varepsilon^{ijk}}{\sqrt{g}}E^\beta_k \kappa^\alpha G_{ij\alpha\beta} \right) ..
\tag{3.9}
$$

Turning to the case of the G_2-membrane this transformation, after insertion of the torsion and field strength, looks like,

$$
\begin{aligned}
\delta_\kappa S &= \int d^3\xi \sqrt{g} \left(E^{\hat\alpha I}_i T_{\hat\alpha I,\hat\beta J}{}^i \kappa^{\hat\beta J} + \frac{1}{2}\frac{\varepsilon^{ijk}}{\sqrt{g}}E^{\hat\alpha I}_k G_{ij\hat\alpha I,\hat\beta J}\kappa^{\hat\beta J} \right) \\
&= \int d^3\xi \sqrt{g}E^{\hat\alpha I}_i \left(2(\gamma^i)_{\hat\alpha\hat\beta} - \frac{\varepsilon^{ijk}}{\sqrt{g}}(\gamma_{jk})_{\hat\alpha\hat\beta} \right) \kappa^{\hat\beta J}\varepsilon_{IJ} \,,
\end{aligned}
\tag{3.10}
$$

which can be rewritten as

$$
\delta_\kappa S = 4 \int d^3\xi \sqrt{g}E^{\hat\alpha I}_i (\gamma^i \Pi_+)_{\hat\alpha\hat\beta}\kappa^{\hat\beta J}\varepsilon_{IJ}
\tag{3.11}
$$

The κ-symmetry condition is thus $(\Pi_+)^{\hat\alpha}{}_{\hat\beta}\kappa^{\hat\beta I} = 0$, where

$$
(\Pi_+)^{\hat\alpha}{}_{\hat\beta} \equiv \frac{1}{2}\left\{ \delta^{\hat\alpha}{}_{\hat\beta} + \frac{1}{6\sqrt{g}}\varepsilon^{ijk}(\gamma_{ijk})^{\hat\alpha}{}_{\hat\beta} \right\}
\tag{3.12}
$$

is the operator which annihilates an infinitesimal κ-parameter. The fact that

$$
\Gamma^{\hat\alpha}{}_{\hat\beta} \equiv \frac{1}{6\sqrt{g}}\varepsilon^{ijk}(\gamma_{ijk})^{\hat\alpha}{}_{\hat\beta} = \frac{1}{6\sqrt{g}}\varepsilon^{ijk}
\begin{bmatrix}
\sigma_{ijk} & -\star\sigma_{ijk\beta} \\
-\star\sigma_{ijk}{}^\alpha & 6\left(\delta^{(\alpha}_{[i}\sigma^{\beta)}_{jk]} - \frac{1}{6}\delta^{\alpha\beta}\sigma_{ijk}\right)
\end{bmatrix}
\tag{3.13}
$$

fulfills the conditions $\mathrm{Tr}(\Gamma) = 0$ and $\Gamma^2 = 1$ implies that Π_+ is a projection operator[2]. It is then obvious that the κ-symmetry condition can be solved by $\kappa = \Pi_-\xi$, where Π_- is defined

---

[2]An essential observation for the working of κ-symmetry is that the euclidean signature of the world-volume is compensated by the fact that the gamma matrices square to minus one. Compared to 11-dimensional Minkowski space there are two changes of sign. Had only one of these changes occurred, idempotent projection matrices could not have been constructed.

as $(\Pi_-)^{\hat{\alpha}}{}_{\hat{\beta}} \equiv \frac{1}{2}(\delta^{\hat{\alpha}}{}_{\hat{\beta}} - \Gamma^{\hat{\alpha}}{}_{\hat{\beta}})$ and ξ is an arbitrary spinor. Since Π_- projects out half of the degrees of freedom of ξ, κ is parametrised by two scalars and two world-volume vectors $\{\lambda^{0I}, \lambda^{iI}\}$. It can be shown that $\Pi_\pm$ are the only projection operators, which project out precisely half of the spinors, that can be formed using the G_2 invariant tensors only, and hence we have found the most general κ-variation.

A byproduct of the above calculation is that the fermionic equations of motion are $\Pi_+ \gamma^i \mathcal{E}_i^{\hat{\alpha}I} = 0$.

A general background will of course contain fermionic excitations, demanding that κ-symmetry is checked also at dimension $\frac{1}{2}$. In the present context, however, we are only interested in superspaces extending any bosonic manifold of G_2 holonomy. We do not consider deformations of the geometry. In topological M-theory, such deformations should be parametrised by solutions of the Hitchin model, and purely bosonic.

The algebra of κ-symmetry is obtained by commuting κ-variations of a fermionic variable, which after some calculation, mainly involving transformation of the projection matrix, yields

$$
\begin{aligned}
[\delta_{\tilde{\kappa}}, \delta_\kappa]\psi^{\hat{\alpha}I} = {} & \varepsilon_{LK}(\Pi_+ \gamma^i \mathcal{E}_i)^{\hat{\alpha}K} \tilde{\kappa}^{(L} \kappa^{I)} - \varepsilon_{LK}(\gamma^j)^{\hat{\alpha}}{}_{\hat{\beta}}(\Pi_+ \gamma^i \mathcal{E}_i)^{\hat{\beta}K} \tilde{\kappa}^{[L} \gamma_j \kappa^{I]} \\
& + (\Pi_-)^{\hat{\alpha}}{}_{\hat{\beta}} \left\{ \frac{1}{2}[(\delta_{\tilde{\kappa}}\Pi_-)\xi - (\delta_\kappa \Pi_-)\tilde{\xi}]^{\hat{\beta}I} + (\mathcal{E}_i)^{\hat{\beta}I} \varepsilon_{KL} \tilde{\kappa}^K \gamma^i \kappa^L \right\} \\
& - (\mathcal{E}_i)^{\hat{\alpha}I} \varepsilon_{KL} \tilde{\kappa}^K \gamma^i \kappa^L .
\end{aligned}
\tag{3.14}
$$

It is straight-forward to see that the three rows represent fermionic equations of motion, κ-transformations and world-volume diffeomorphisms, respectively. This is the point where it becomes clear that the formulation, due to the mismatch between fermions and bosons off-shell, is an on-shell formulation — part of the gauge symmetry only works modulo equations of motion.

Although we will not elaborate on this in the present paper, it is worth mentioning that κ-symmetry can in fact be treated in a completely covariant manner on a G_2 manifold. The projection $\kappa = \Pi_+ \kappa$ may be solved by parametrising κ in terms of a scalar and a world-volume vector as

$$
\begin{aligned}
\kappa^0 &= (1-y)\xi \,, \\
\kappa^\alpha &= z^\alpha \xi + (E_i^\alpha - \tfrac{1}{2\sqrt{g}}\varepsilon_i{}^{jk}\sigma^\alpha{}_{jk})\zeta^i \,,
\end{aligned}
\tag{3.15}
$$

where $y = \frac{1}{6\sqrt{g}}\varepsilon^{ijk}\sigma_{ijk}$, $z_\alpha = \frac{1}{6\sqrt{g}}\varepsilon^{ijk}\star\sigma_{ijk\alpha}$. In a situation where the scalar part has been fixed, the remaining gauge symmetry (closing on-shell) will be a super-diffeomorphism algebra with an SL(2) doublet of world-volume vectors as fermionic generators.

There is an interplay between the global supersymmetry and the local κ-symmetry, in the sense that both transform the singlet fermions θ^I. Even if the supersymmetry generators obey eq. (2.19) exactly and without reference to the embedding of the membrane world-volume, this ceased to be true once κ-symmetry is gauge fixed. When some gauge is chosen that involves θ^I (which any gauge has to), compensating gauge transformations have to be introduced in order that the redefined supersymmetry generator transforms within the constraint surface defined by the gauge choice. Then, due to the commutation

relation (3.14), the nilpotency relation (2.19) only holds on-shell, i.e., modulo fermionic equations of motion.

3.3 Topological membranes

In order to restrict the supermembrane theory to a topological theory, we want to promote the two supercharges Q_I to BRST operators, and let the theory be defined by cohomology of these. Unlike the theory of topological strings, where one in a conformal gauge has a split in left- and right-movers (or holomorphic and anti-holomorphic dependence of the world-sheet coordinate), there is no such natural split, and one has to treat the two supersymmetry generators simultaneously.

We have already shown how the invariance of the supermembrane action works. If the theory is to become a cohomological field theory, it is important that the action not only is invariant, but also trivial in cohomology, i.e., BRST-exact. This means that there should exist a functional Σ^I with

$$Q_I \Sigma^J = \delta_I^J S \,. \tag{3.16}$$

The simple form of the supersymmetry generators assures that this is achieved by $\Sigma^I = \int \mathcal{L} \theta^I$, if $Q_I \mathcal{L} = 0$ locally on the world-volume, and the action is then invariant without resort to partial integration. We have demonstrated earlier that this is actually the case, due to the fact that a gauge can be chosen where the 3-form C is independent of θ. The proof of this statement involved the explicit construction of the superspace 3-form, which used flat space expressions, but should be possible to generalise.

The "pre-action" Σ^I is defined modulo Q-exact terms, encoded in $Q_I \Xi^{JK} = \delta_I^K \Delta \Sigma^J$, which can be seen as "gauge transformations" in the complex. It is important that other gauge symmetries in the model are consistent with this one, in the sense that Σ^I must be invariant modulo terms of this trivial type. This applies especially to κ-symmetry, which is not manifest. Indeed, the fact that the κ-variation of $\mathcal{L}$ is a total derivative ensures that, with the above form of Σ^I, $\delta_\kappa \Sigma^I$ is trivial. This property becomes essential e.g. when one wants to perform a gauge-fixing of a part of κ-symmetry that transforms θ^I. Then, Q_I has to be supplemented with a compensating gauge transformation, which can not be allowed to interfere with cohomology. Consider an infinitesimal "deformation" of the supersymmetry generator by a κ-transformation, $\tilde{Q}_I = Q_I + M_I^{\mathcal{A}} t_{\mathcal{A}}$, where $t_{\mathcal{A}}$ are generators of some gauge transformations labelled by the index $\mathcal{A}$, and $M_I^{\mathcal{A}}$ are infinitesimal parameters. If $Q_I \Xi_{\mathcal{A}}^{JK} = \frac{1}{2} \delta_I^K Q_L \Xi_{\mathcal{A}}^{JL} = \delta_I^K t_{\mathcal{A}} \Sigma^J$ as above, one can define $\tilde{\Sigma}^I = \Sigma^I - M_J^{\mathcal{A}} \Xi_{\mathcal{A}}^{IJ}$, and still have $\tilde{Q}_I \tilde{\Sigma}^J = \delta_I^J S$. A finite deformation, as when gauge-fixing is performed, will require the discussion to be extended to an infinite sequence of descent equations.

An interesting parallel to topological string theory can be observed when one tries to construct a Σ^I that is "as κ-invariant as possible", order by order in fermions. An Ansatz would, apart from the expression above, include terms that are independent of θ,

$$\Sigma^I = \int d^3\xi \sqrt{g} \theta^I + \int ((C + k\Omega)\theta^I + R^I) \,. \tag{3.17}$$

Here, R is a 3-form with $Q_I R = 0$, and the the term containing Ω modifies eq. (3.16) with a purely topological term,

$$Q_I \Sigma^J = \delta_I^J (S + k\Omega).\tag{3.18}$$

Using elements of the calculation yielding κ-symmetry of the action, one finds

$$\delta_\kappa \Sigma^I = \int i_\kappa((\star 1 + C + k\Omega)\wedge d\theta^I + dR^I).\tag{3.19}$$

Invariance at lowest order can be achieved if $k = 1$ and $R_{abc}^I = -\star\sigma_{abc\alpha}\psi^{\alpha I}$, in which case the lowest order variation becomes $\int 2(\Pi_+ \kappa^I)^0 = 0$, which is seen from the decomposition (3.13) of the projection matrix in G_2 tensors. However, exact cancellation to all orders is not possible by addition of further terms in R^I. Again, of course, the non-zero terms in the variation are trivial. The relation (3.18), with $k = 1$, is the exact correspondence to the fact that in topological string theory, the BRST-trivial object is the action plus the integral of the Kähler form, which is obtained from Ω on dimensional reduction.

4. Topological membranes in topological M-theory

We have shown how a supermembrane in seven dimensions with euclidean signature can be turned into a topological theory. It would be interesting to study the quantum mechanical properties of the topological membrane theory, and investigate to what extent the quantum theory reproduces topological M-theory. The best framework for doing this would be one including a proper set of auxiliary fields that makes the symmetries of the theory valid of-shell. It seems much harder to reach such a formulation in the present situation than for the usual world-sheet supersymmetric sigma model on which topological string theory is based.

It is clear that associative cycles [17] are solutions of the theory. These are calibrating cycles for the 3-form Ω. An easy way to see that associative cycles are supersymmetric is to partially fix gauge for κ-symmetry by demanding $\theta^I = 0$. The supersymmetry, including a compensating κ-transformation, on the remaining fermions becomes

$$\delta_\varepsilon \psi^{\alpha I} = -\frac{z^\alpha}{1 - y}\varepsilon^I,\tag{4.1}$$

where $y = \frac{1}{6\sqrt{g}}\varepsilon^{ijk}\sigma_{ijk}$, $z_\alpha = \frac{1}{6\sqrt{g}}\varepsilon^{ijk}\star\sigma_{ijk\alpha}$. A configuration is supersymmetric if $z^\alpha = 0$, giving the possibilities $y = \pm 1$, and if eq. (4.1) is to be well behaved only $y = -1$ is possible. With a non-zero Wess-Zumino term in the membrane action we are actually dealing with a generalised calibration, see e.g. refs. [18, 19]. It is however of a trivial type since the bosonic 3-form is closed and hence the Wess-Zumino term contributes equally to all cycles minimal or not.

Looking for local observables seems more problematic. In the A-model, considering collapsed, point-like, world-sheets is straightforward, and cohomology of the BRST-operator is directly translated into cohomology for a de Rahm-complex for the Calabi-Yau manifold. In the present situation, we have to take κ-symmetry into account, with its projection that depends on the orientation of the embedded world-volume. We have not yet been able to

address this question in a constructive way, and thus can not present a direct connection between observables for the topological membrane and Hitchin's theory.

It is clear that a double dimensional reduction of the topological membrane produces the strings of the topological A-model, although formulated in a space supersymmetric rather than world-sheet supersymmetric way. Associative cycles will map to holomorphic cycles. For the same reasons as above, we are not able to make a corresponding statement concerning local observables (although investigating this question for the A-model starting from a Green-Schwarz formulation might give some insight)[3]. A direct reduction will of course give A-model 2-branes. These are not D-branes. An A-model D2-brane must be represented by a 3-brane in $D = 7$, since the boundary of an open membrane winding the compactified circle also winds. This, along with the existence of the dual form $\star\Omega$, makes it clear that 3-branes, on which the membranes may end, are needed in topological M-theory. The 3-branes, living on the same superspace, should support a world-volume 2-form potential, with a 3-form field strength. This field, that can be dualised to a scalar, accounts for the correct matching of bosonic and fermionic degrees of freedom. An interesting observation, on which we would like to elaborate in the future, is that although $(\gamma_{abc})_{\hat\alpha\hat\beta}$ is symmetric in spinor indices, and thus cannot be used in a dimension-0 component of the 5-form field strength for the 4-form potential coupling to the 3-brane, there exists a closed 5-form constructed from G_2-invariant tensors.

It would be a great step forward to find a good set of auxiliary fields for the membrane theory, that would allow for an off-shell formulation, and hopefully make quantisation more manageable. Although this, in general backgrounds, would probably be to ask too much, it is maybe not unrealistic to hope that the G_2 structure would help. It turns spinors into scalars and vectors, and even κ-symmetry can be parametrised covariantly, as in eq. (3.15). One possible starting point could be the construction of a super-diffeomorphism algebra on the world-volume containing an SL(2) doublet of fermionic vector generators, similar to what one obtains after gauge-fixing the scalar part of κ-symmetry.

Although we do not claim to have a microscopic definition of topological M-theory, we hope that the present work represents a step in that direction. Maybe it can be a point of departure for a refined formulation, where urgent questions, such as the connection to Hitchin's theory of G_2 moduli, can be answered. Such a formulation might also give valuable insight into the question of how membrane functional integrals are performed (see e.g. the discussion in ref. [7]). Earlier experience of instanton counting on compact submanifolds have shown that naive counting of membrane configurations may lead to incorrect results [24, 25], and a proper theory of topological membranes may be a place where such issues can be addressed in a precise manner.

A. Some details on G_2 tensors

We use e.g. the expressions $\sigma_{a,a+1,a+3} = 1$ (where indices are counted modulo 7), giving $\star\sigma_{a,a+1,a+2,a+5} = 1$. $\star\sigma$ is the octonionic associator, $[e_a, e_b, e_c] = (e_a e_b)e_c - e_a(e_b e_c) =$

---

[3]Such a formulation will be possible directly in six dimensions for both the A- and B-models. One has a priori an SL(2) doublet of complex supersymmetries, of which different real combinations may be chosen.

$-2\star\sigma_{abcd}e_d$. Useful relations between octonionic structure constants:

$$\begin{aligned}
\sigma_{acd}\sigma_b{}^{cd} &= 6\delta_{ab}\,,\\
\star\sigma_{abde}\sigma_c{}^{de} &= -4\sigma_{abc}\,,\\
\sigma_{abc}\sigma^{abc} &= 42\,,\\
\sigma_{abe}\sigma_{cd}{}^e &= 2\delta_{cd}^{ab} - \star\sigma_{abcd}\,,\\
\sigma_{abf}\star\sigma_{cde}{}^f &= 6\delta_{[c}^{[a}\sigma^{b]}{}_{de]}\,,\\
\star\sigma_{abcg}\star\sigma_{def}{}^g &= 6\delta_{def}^{abc} - 3\delta_{[d}^{[a}\star\sigma^{bc]}{}_{ef]} - 3\sigma_{[ab}{}^{[d}\sigma_{c]}{}^{ef]}\,,\\
\star\sigma_{abef}\star\sigma_{cd}{}^{ef} &= 8\delta_{cd}^{ab} - 2\star\sigma_{abcd}\,.
\end{aligned}$$
(A.1)

The last of these relations can be used to find projections on the 7- and 14-dimensional vector spaces in $21 \to 14 \oplus 7$ under $\mathrm{Spin}(7) \to G_2$ as

$$\begin{aligned}
\Pi^{(14)}{}_{ab}{}^{cd} &= \frac{2}{3}(\delta_{ab}^{cd} + \frac{1}{4}\star\sigma_{ab}{}^{cd})\,,\\
\Pi^{(7)}{}_{ab}{}^{cd} &= \frac{1}{3}(\delta_{ab}^{cd} - \frac{1}{2}\star\sigma_{ab}{}^{cd})\,.
\end{aligned}$$
(A.2)

It can be noted that σ_{abc}, seen as a set of seven matrices $(\sigma_a)_{bc}$, are in the 7-dimensional subspace: $(\Pi^{(14)}\sigma_a)_{bc} = 0$, and actually provide a basis for it.

Consider the split of the octonions $\mathbb{O}$ as $\mathbb{H} \oplus \mathbb{H}$, and write $x = \xi + j\eta$, where ξ and η are quaternions and j is an imaginary unit orthogonal to $\mathbb{H}$. The octonionic multiplication is encoded in terms of the quaternions by the multiplication rules $ja = a^*j$, $(ja)b = j(ba)$ for all $a, b \in \mathbb{H}$. Then

$$xx' = \xi\xi' - \eta'\eta^* + j(\xi^*\eta' + \xi'\eta)\,.$$
(A.3)

We want to examine which of the rotations in $\mathrm{SO}(3) \times \mathrm{SO}(4)$ acting on imaginary octonions and preserving this split are automorphisms, i.e., belong to G_2. The rotations are parametrised as $\xi \to \sigma^*\xi\sigma$, $\eta \to e^*\eta e'$, where all three parameters are unit quaternions. A direct check with eq. (A.3) yields that the necessary condition for this to be an automorphism is $\sigma = e$, verifying that the common subgroup of this $\mathrm{SO}(3) \times \mathrm{SO}(4)$ and G_2 is $\mathrm{SU}(2) \times \mathrm{SU}(2)$, and that the twisting — the identification of world-volume $\mathrm{SO}(3)$ rotations with a transverse $\mathrm{SU}(2)$—takes place.

The remaining part of the G_2 algebra transforms as $(4, 2)$, and is realised infinitesimally with a "vector-spinor" h_i, $i = 1, 2, 3$, in $\mathbb{H}^{\otimes 3}$ with $e_i h_i = 0$. The transformations are $\delta\xi = e_i\eta h_i^*$, $\delta\eta = e_i\xi h_i \,(= -2\xi_i h_i)$, and the derivation property may be checked explicitly.

The split into two quaternions can also be seen as a split in four complex numbers with imaginary unit j. With $x = z_0 + z^i e_i$, the multiplication table is $xx' = z_0 z_0' - z^i \bar{z}_i' + (z_0 z'^i + z^i \bar{z}_0' + \varepsilon^{ijk}\bar{z}_j\bar{z}_k')e_i$, in which $\mathrm{SU}(3) \subset G_2$ is a manifest automorphism. The rest of the automorphisms are parametrised by λ^i, $\bar{\lambda}_i$ in $3 \oplus \bar{3}$, acting as $\delta z_0 = \lambda^i\bar{z}_i - \bar{\lambda}_i z^i$, $\delta z^i = \lambda^i(z_0 - \bar{z}_0) + \varepsilon^{ijk}\bar{\lambda}_j\bar{z}_k$.

B. 3-forms in superspace

The field strength G is related to the potential C in the conventional way

$$G = dC \quad \Rightarrow \quad G_{ABCD} = 4\delta_{[A}C_{BCD)} + 6T_{[AB}{}^F C_{|F|CD)}\,,$$
(B.1)

where the indices in capital letters are the entire superspace indices. The bracket $[*]$ denotes a graded symmetrisation. Using the fact that in a flat background, the only non-vanishing components of G_{ABCD} and $T_{AB}{}^C$ are $G_{ab,\hat\gamma I,\hat\delta J} = -2\varepsilon_{IJ}(\gamma_{ab})_{\hat\gamma\hat\delta}$ and $T_{\hat\alpha I,\hat\beta J}{}^c = 2\varepsilon_{IJ}(\gamma^c)_{\hat\alpha\hat\beta}$, respectively, the equations (B.1) can be solved for C_{ABC}. The solution we are interested in has the property that the only coordinate dependence is through the seven-dimensional fermionic coordinates $\psi^{\alpha I}$. By looking at the group representation structures of the different components of C, we make an Ansatz for the potential using G_2 invariants. Due to invariance under the gauge transformations $\delta C = d\Lambda$, some of the parameters in the Ansatz are free, and set to zero for simplicity. We use flat space or work to lowest order in curvatures. The potential we have found can be written as $C^{(k)} = C + k\Omega$, where

$$C_{abc} = 0$$
$$C_{ab,0I} = \varepsilon_{IL}\psi^{\delta L}2\sigma_{ab\delta}$$
$$C_{ab,\alpha I} = \varepsilon_{IL}\psi^{\delta L}(2\delta^{ab}_{\alpha\delta} + \star\sigma_{ab\alpha\delta})$$
$$C_{a,0I,0J} = \varepsilon_{I(L}\varepsilon_{|J|M)}\psi^{\delta L}\psi^{\epsilon M}(-4)\sigma_{a\delta\epsilon}$$
$$C_{a,\alpha I,0J} = \varepsilon_{I(L}\varepsilon_{|J|M)}\psi^{\delta L}\psi^{\epsilon M}\left(-4\delta^{a\alpha}_{\delta\epsilon} - \frac{2}{3}\star\sigma_{a\alpha\delta\epsilon}\right)$$
$$C_{a,\alpha I,\beta J} = \varepsilon_{I(L}\varepsilon_{|J|M)}\psi^{\delta L}\psi^{\epsilon M}\left(-\frac{4}{3}\sigma_{a\delta\epsilon}\delta_{\alpha\beta} - \frac{8}{3}\sigma_{a\alpha\delta}\delta_{\beta\epsilon}\right)$$
$$C_{0I,0J,0K} = \varepsilon_{I(L}\varepsilon_{|J|M}\varepsilon_{|K|N)}\psi^{\delta L}\psi^{\epsilon M}\psi^{\phi N}8\sigma_{\delta\epsilon\phi}$$
$$C_{\alpha I,0J,0K} = \varepsilon_{IL}\varepsilon_{J(M}\varepsilon_{|K|N)}\psi^{\delta L}\psi^{\epsilon M}\psi^{\phi N}\left(-\frac{8}{3}\right)\star\sigma_{\alpha\delta\epsilon\phi}$$
$$C_{\alpha I,\beta J,0K} = \varepsilon_{IL}\varepsilon_{JM}\varepsilon_{KN}\psi^{\delta L}\psi^{\epsilon M}\psi^{\phi N}\left(\frac{8}{3}\delta_{\alpha\beta}\sigma_{\delta\epsilon\phi} + 8\sigma_{\alpha\epsilon\phi}\delta_{\beta\delta}\right)$$
$$C_{\alpha I,\beta J,\gamma K} = \varepsilon_{IL}\varepsilon_{JM}\varepsilon_{KN}\psi^{\delta L}\psi^{\epsilon M}\psi^{\phi N}8\delta_{\alpha\phi}\star\sigma_{\beta\gamma\delta\epsilon} \tag{B.2}$$

$$\Omega_{abc} = \sigma_{abc}$$
$$\Omega_{ab,0I} = \varepsilon_{IL}\psi^{\delta L}(-2)\sigma_{ab\delta}$$
$$\Omega_{ab,\alpha I} = \varepsilon_{IL}\psi^{\delta L}(-2\delta^{ab}_{\alpha\delta} + \star\sigma_{ab\alpha\delta})$$
$$\Omega_{a,0I,0J} = \varepsilon_{I(L}\varepsilon_{|J|M)}\psi^{\delta L}\psi^{\epsilon M}4\sigma_{a\delta\epsilon}$$
$$\Omega_{a,\alpha I,0J} = \varepsilon_{I(L}\varepsilon_{|J|M)}\psi^{\delta L}\psi^{\epsilon M}(4\delta^{a\alpha}_{\delta\epsilon} + 2\star\sigma_{a\alpha\delta\epsilon})$$
$$\Omega_{a,\alpha I,\beta J} = \varepsilon_{I(L}\varepsilon_{|J|M)}\psi^{\delta L}\psi^{\epsilon M}\left(\frac{8}{3}\delta_{a\alpha}\sigma_{\beta\delta\epsilon} - \frac{4}{3}\sigma_{a\delta\epsilon}\delta_{\alpha\beta} - \frac{8}{3}\sigma_{a\alpha\delta}\delta_{\beta\epsilon}\right)$$
$$\Omega_{0I,0J,0K} = \varepsilon_{I(L}\varepsilon_{|J|M}\varepsilon_{|K|N)}\psi^{\delta L}\psi^{\epsilon M}\psi^{\phi N}(-8)\sigma_{\delta\epsilon\phi}$$
$$\Omega_{\alpha I,0J,0K} = \varepsilon_{IL}\varepsilon_{J(M}\varepsilon_{|K|N)}\psi^{\delta L}\psi^{\epsilon M}\psi^{\phi N}4\star\sigma_{\alpha\delta\epsilon\phi}$$
$$\Omega_{\alpha I,\beta J,0K} = \varepsilon_{IL}\varepsilon_{JM}\varepsilon_{KN}\psi^{\delta L}\psi^{\epsilon M}\psi^{\phi N}\left(-\frac{8}{3}\delta_{\alpha\beta}\sigma_{\delta\epsilon\phi} - \frac{56}{3}\sigma_{\alpha\epsilon\phi}\delta_{\beta\delta}\right)$$
$$\Omega_{\alpha I,\beta J,\gamma K} = \varepsilon_{IL}\varepsilon_{JM}\varepsilon_{KN}\psi^{\delta L}\psi^{\epsilon M}\psi^{\phi N}8\delta_{\alpha\phi}\star\sigma_{\beta\gamma\delta\epsilon} \tag{B.3}$$

and k is a free parameter. Symmetrisation in composite fermionic indices is implicitly understood in eqs. (B.2) and (B.3). Eq. (B.3) can of course be obtained directly (modulo

an exact form) by expanding the bosonic differentials in $\Omega = \frac{1}{6} dx^p \wedge dx^n \wedge dx^m e_p{}^c e_n{}^b e_m{}^a \sigma_{abc}$ using the vielbeins of eq. (2.16).

The fact that θ^I are G_2-invariant should make it clear that the proof of local θ-independence of the lagrangian, on which the BRST-exactness relies, may be generalised to curved backgrounds, involving modifications of the explicit forms of the supervielbeins of eq. (2.16) and the super-3-forms of eqs. (B.2) and (B.3).

Acknowledgments

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